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Dilute Bose gas in two dimensions: Density expansions and the Gross-Pitaevskii equation

A. Yu. Cherny and A. A. Shanenko
Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141980, Dubna, Moscow Region, Russia

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A dilute homogeneous two-dimensional (2D) Bose gas at zero temperature is studied with the method developed earlier by the authors. This method allows for considering renormalization of an arbitrary pairwise potential in a self-consistent manner, without the pseudopotential δ-function representation. Low-density expansions are derived for the chemical potential, ground-state energy, pair distribution function, kinetic and interaction energies. The expansion parameter is found to be a dimensionless in-medium scattering amplitude u obeying the equation $1/u+\ln u=-\ln(na^2\pi)-2\gamma$, where na^2 and γ are the gas parameter and the Euler constant, respectively. It is shown that the ground-state energy is mostly kinetic in the low-density limit. This result does not depend on a specific form of the pairwise interaction potential, contrary to the 3D case. A new form of the 2D Gross-Pitaevskii equation is proposed within our scheme.

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Theoretical investigation of two-dimensional (2D) Bose gas is of interest not only in itself but also from the point of view of its possible experimental applications. Indeed, the experimental observation of the Bose-Einstein quasicondensate of hydrogen atoms has been reported [1]. On the other hand, the discovery of the Bose-Einstein condensation in magnetically trapped alkali-metal atoms [2] stimulated rapid progress in optical cooling and trapping of atoms. This progress gives us hope that an experimental observation of 2D quasicondensation in the trapped atoms is a matter of time [3]. Theoretically, the Bose-Einstein condensation is associated with the off-diagonal long-range order, i.e., the nonzero asymptotic at $r=|\mathbf{r}_1-\mathbf{r}_2|\to\infty$ (from the physical point of view at $r\gg 1/\sqrt{n}$) for the one-body density matrix

$$\langle\hat{\psi}^{\dagger}(\mathbf{r}_1)\hat{\psi}(\mathbf{r}_2)\rangle\to\langle\hat{\psi}^{\dagger}(\mathbf{r}_1)\rangle\langle\hat{\psi}(\mathbf{r}_2)\rangle\neq 0. \tag{1}$$

Here, $\hat{\psi}^{\dagger}(\mathbf{r})$ and $\hat{\psi}(\mathbf{r})$ are the Bose field operators, $\langle\cdots\rangle$ stands for the statistical average, and $\langle\hat{\psi}(\mathbf{r})\rangle=\phi(\mathbf{r})$ is the order parameter. As was shown by Hohenberg [4] from the Bogoliubov "$1/q^2$" theorem [5], in the 2D case, there is no off-diagonal long-range order at finite temperatures due to the temperature long-range fluctuations of the phase, and the limit (1) is equal to zero. In spite of this fact, a phase transition is possible to a superfluid state at sufficiently low temperature T_c [6]. At this temperature, the asymptotic behavior of the one-body density matrix at $r\to\infty$ is changed from an exponential decay (above T_c) to a power decay (below T_c) with respect to r, the phenomenon of the quasicondensation in two dimensions. For zero temperature, the limit (1) differs from zero, and, hence, there exists the true Bose-Einstein condensate. Assuming that the condensate does exist at $T=0$, in this paper, we consider low-density expansions for 2D homogeneous Bose gas with respect to the gas parameter na^2, where $n=N/S$ is the density (a number of particles per unit area), and a stands for the *two-dimensional* scattering length (see the Appendix). Note that the density expansion for the chemical potential is intimately related to a form of the Gross-Pitaevskii equation, a powerful tool for investigating a dilute inhomogeneous system of Bose particles [7]. As the chemical potential is a continuous function of temperature, one can expect that our results, obtained for zero temperature, are also valid at least for the temperatures $T\ll T_c$. Below we consider the case of zero temperature only.

The leading term of the energy expansion in na^2 for a gas of hard discs was first obtained by Schick [8], who made use of the Beliaev method [9], developed for 3D Bose gas. Recently, the Schick asymptotic formula has been proved rigorously [10]. To the best of our knowledge, there is only one paper, by Hines *et al.* [11], where four next-to-leading terms were evaluated [12]. However, the authors of Ref. [11] employed the *first-order* Beliaev approximation for obtaining the *next-to-leading* terms, while in three dimensions, the *second-order* Beliaev approximation is needed for the same purpose. Following Schick, they ignored the imaginary part of the Beliaev equation for the chemical potential [compare Eq. (4.3) in Ref. [9] with Eq. (1) in Ref. [11]]. It is the unphysical imaginary correction to the chemical potential that determines the range of validity of the first-order approximation itself. One can easily demonstrate that in the 2D case the correction is of the order of $n/\ln^2(na^2)$; therefore, that method cannot yield correct terms of the expansion in this order and higher. At the same time, our method [13] successfully reproduces in the 3D case the famous next-to-leading term for the chemical potential, which ensures that our results are valid also in two dimensions. Thus, it becomes clear why only the first two terms involved in our expansion [see Eqs. (18), (24) below] coincide with those of the corresponding expansion from Ref. [11]: the correct coefficient even for the third term is beyond the approximation used in Ref. [11].

In this paper, we adopt the method developed and described in detail in our previous publications [13]. Only basic notations and some important points are discussed here. For a homogeneous system, the one-body density matrix $\langle\hat{\psi}^{\dagger}(\mathbf{r}_1)\hat{\psi}(\mathbf{r}_2)\rangle$ depends on $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$, and, hence, its eigenfunctions and eigenvalues are the plane waves $\exp(i\mathbf{p}\cdot\mathbf{r})/\sqrt{S}$ and the occupation numbers $n_p=\langle\hat{a}^{\dagger}_{\mathbf{p}}\hat{a}_{\mathbf{p}}\rangle$, respectively. The Bose-Einstein condensate corresponds to the macroscopic occupation number N_0, and the order parameter is $\langle\hat{\psi}(\mathbf{r})\rangle=\langle\hat{a}_0\rangle/\sqrt{S}=\sqrt{n_0}e^{i\chi}$ (n_0 stands for the density of the condensate). In turn, the eigenfunctions of the two-body density matrix $\langle \hat{\psi}^{\dagger}(\mathbf{r}_1)\hat{\psi}^{\dagger}(\mathbf{r}_2)\hat{\psi}(\mathbf{r}_2')\hat{\psi}(\mathbf{r}_1')\rangle$ can be naturally classified as follows. The maximum eigenvalue $N_0(N_0-1) \sim N_0^2$ corresponds to the state of two particles in the condensate; its eigenfunction $\varphi(r)/S$ can be interpreted as a pair wave function *in medium* of the condensate-condensate type. The other macroscopic eigenvalues $2N_0 n_q$ correspond to the two-body states with one particle in the condensate and another one beyond the condensate; its eigenfunctions $\varphi_{\mathbf{q}/2}(\mathbf{r})\exp[i\mathbf{q}\cdot(\mathbf{r}_1+\mathbf{r}_2)/2]/S$ are of the condensate-supracondensate type, where $\hbar\mathbf{q}$ is the total momentum of the pair of bosons. The residuary nonmacroscopic eigenvalues are related to the supracondensate-supracondensate pairs and to bound ones provided the latter exist. The functions $\varphi(r)$ and $\varphi_{\mathbf{q}/2}(\mathbf{r})$ can be chosen as real quantities given by

$$\varphi(r)=1+\psi(r), \quad \varphi_{\mathbf{p}}(\mathbf{r})=\sqrt{2}\cos(\mathbf{p}\cdot\mathbf{r})+\psi_{\mathbf{p}}(\mathbf{r}) \quad (p\neq 0) \tag{2}$$

with the boundary conditions $\psi(r)$, $\psi_{\mathbf{p}}(\mathbf{r})\to 0$ at $r\to\infty$. The Fourier transforms of the scattering parts can be expressed in terms of the Bose operators:

$$\psi(k)=\frac{\langle \hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}}\rangle}{n_0}, \quad \psi_{\mathbf{p}}(\mathbf{k})=\sqrt{\frac{S}{2n_0}}\frac{\langle \hat{a}^{\dagger}_{2\mathbf{p}}\hat{a}_{\mathbf{p}+\mathbf{k}}\hat{a}_{\mathbf{p}-\mathbf{k}}\rangle}{n_{2p}}. \tag{3}$$

With the help of the in-medium scattering amplitudes $U(k)=\int d^2r\,\varphi(r)V(r)\exp[-i\mathbf{k}\cdot\mathbf{r}]$ and $U_{\mathbf{p}}(\mathbf{k})=\int d^2r\,\varphi_{\mathbf{p}}(\mathbf{r})V(r)\exp[-i\mathbf{k}\cdot\mathbf{r}]$, the chemical potential reads

$$\mu=n_0U(0)+\sqrt{2}\int\frac{d^2q}{(2\pi)^2}n_qU_{\mathbf{q}/2}(\mathbf{q}/2). \tag{4}$$

Here we introduce a pairwise interaction potential $V(r)$. It should be emphasized that the formulas (3) and (4), derived [13] within the Bogoliubov principle of the correlation weakening [5], are *exact*. For a Bose gas, a system with a small condensate depletion $(n-n_0)/n \ll 1$, the pair distribution function is expressed as

$$g(r)=\left(\frac{n_0}{n}\right)^2\varphi^2(r)+2\frac{n_0}{n}\int\frac{d^2q}{(2\pi)^2}\frac{n_q}{n}\varphi^2_{\mathbf{q}/2}(\mathbf{r}), \tag{5}$$

where the contribution of the supracondensate-supracondensate pair wave functions can be neglected; another restriction is the assumption that there are no bound pair states in the medium [13]. In order to fulfil the latter condition, it is sufficient to require $V(r)>0$, and, as usually, $V(r)\to 0$ for $r\to\infty$. In the framework of our scheme, the following equations are valid *at sufficiently low densities* [13]:

$$n_k=\frac{1}{2}\left(\frac{T_k+nU(k)}{\sqrt{T_k^2+2nT_kU(k)}}-1\right), \tag{6}$$

$$\psi(k)=-\frac{1}{2}\frac{U(k)}{\sqrt{T_k^2+2nT_kU(k)}}, \tag{7}$$

with $T_k=\hbar^2k^2/(2m)$. Equations (6) and (7) look like those of the modified Bogoliubov model where the "bare" pairwise potential $V(k)$ is replaced by the effective one $U(k)$ that is determined from the two-body Schrödinger equation. However, in our method, there exists a key difference, which is of particular importance in two dimensions: Eq. (7) is a *self-consistent* equation for the in-medium scattering amplitude $U(k)$. Indeed, using the definition of the latter, Eq. (7) can be represented in the Lippmann-Schwinger form

$$U(k)=V(k)-\frac{1}{2}\int\frac{d^2q}{(2\pi)^2}\frac{V(|\mathbf{k}-\mathbf{q}|)U(q)}{\sqrt{T_q^2+2nT_qU(q)}}. \tag{8}$$

Besides, one can make use of the limiting relation [see Eq. (2)] $\lim_{p\to 0}\varphi_{\mathbf{p}}(\mathbf{r})=\sqrt{2}\varphi(r)$, which allows one to simplify Eqs. (4) and (5)

$$\mu=nU(0)[1+(n-n_0)/n+\cdots], \tag{9}$$

$$g(r)=\varphi^2(r)[1+2(n-n_0)/n+\cdots]. \tag{10}$$

Thus, our scheme is reduced to the following. First, one should solve Eq. (8) and find $U(0)$ [and $\varphi(r)$] as a function of the density at $n\to 0$. Second, the condensate depletion $(n-n_0)/n$ should be determined from Eq. (6). Third, employing these results, one should obtain the density expansion for the chemical potential (9) and the short-range behavior for the pair distribution function (10). Note that in three dimensions, this scheme is in excellent agreement with the data of Monte Carlo calculations for hard spheres (see the last paper in Ref. [13]).

In order to solve Eq. (8) at $n\to 0$, we employ the procedure of linearization, which is similar to that in the 3D case [13], and rewrite this equation in the form

$$U(k)=V(k)-\frac{1}{2}\mathrm{P.P.}\int\frac{d^2q}{(2\pi)^2}\frac{V(|\mathbf{k}-\mathbf{q}|)U(q)}{T_q-T_{q_0}}-\frac{I}{2}, \tag{11}$$

where P.P. denotes the Cauchy principal value, and

$$I=\mathrm{P.P.}\int\frac{d^2q}{(2\pi)^2}\left[\frac{V(|\mathbf{k}-\mathbf{q}|)U(q)}{\sqrt{T_q^2+2nT_qU(q)}}-\frac{V(|\mathbf{k}-\mathbf{q}|)U(q)}{T_q-T_{q_0}}\right].$$

Here, we introduce the auxiliary quantity $q_0=c\sqrt{2mnU(0)}/\hbar$, where c stands for an arbitrary dimensionless constant. Performing the "scaling" substitution

$$\mathbf{q}=\mathbf{q}'\sqrt{2mnU(0)}/\hbar \tag{12}$$

in the integral [$U(0)$ is assumed to depend on n in such a manner that $nU(0)\to 0$ when $n\to 0$] and, then, taking the zero-density limit in the integrand, for $n\to 0$, we find

$$I=2\Lambda V(k), \quad \Lambda=\ln(2c^2)mU(0)/(2\pi\hbar^2). \tag{13}$$

With the help of the Fourier transformation, Eq. (11) reads

$$\varphi(r)=1-\Lambda+\frac{m}{4\hbar^2}\int d^2r'\,V(r')\varphi(r')Y_0(q_0|\mathbf{r}-\mathbf{r}'|).$$

Here, the Fourier representation is used $Y_0(pr) = 4/(2\pi)^2 \mathrm{P.P.}\int \mathrm{d}^2q \exp[i\mathbf{q}\cdot\mathbf{r}]/(p^2-q^2)$ for the cylindrical Bessel function of the second kind. Since only the asymptotic of $\varphi(r)$ at $n\to 0$ is of interest, the linear integral equation for $\varphi(r)$ can be written with the asymptotic $Y_0(z) = 2\ln(ze^{\gamma}/2)/\pi + \mathrm{O}(z^2\ln z)$, valid for $z\to 0$. Here $\gamma \simeq 0.5772$ stands for the Euler constant, and $\mathrm{O}(x)$ denotes terms of the order of x or even higher. It is seen from the resulted equation that, first, $\varphi(r)$ obeys the Schrödinger Eq. (A1), and, second, its asymptotic for $r\to\infty$ is

$$\varphi(r)\to 1-\Lambda+\ln(rq_0e^{\gamma}/2)mU(0)/(2\pi\hbar^2), \tag{14}$$

which differs from that of Eq. (A2) only by the multiplication factor $mU(0)/(2\pi\hbar^2)$. Comparing Eq. (14) with Eq. (A2) yields due to linearity of Eq. (A1)

$$\varphi(r) = 2u\varphi^{(0)}(r), \tag{15}$$

$$-\ln a = 2\pi\hbar^2(1-\Lambda)/[mU(0)] + \ln(q_0e^{\gamma}/2), \tag{16}$$

where we introduce the parameter u by the definition

$$U(0) = \int d^2r\varphi(r)V(r) = (4\pi\hbar^2/m)u. \tag{17}$$

With the help of Eq. (13) and the definition of q_0 (see above), Eq. (16) can be rewritten as

$$u = \delta(1+u\ln u), \quad \delta = -1/[\ln(na^2\pi)+2\gamma]. \tag{18}$$

Note that $\delta\to 0$ at $n\to 0$. As expected, the arbitrary constant c is canceled and not involved in final Eq. (18) and, hence, in the formula (15) for $\varphi(r)$. Equation (18) has no solution for u when $\delta>1$ and has two positive ones when $\delta<1$ (i.e., when $na^2<0.0369\ldots$). The solution with a greater value of u should be ignored because of its unphysical behavior [$u \sim 1/(na^2)$ at $n\to 0$]. An expansion for u is obtained from Eq. (18) by iterations

$$u = \delta + \delta^2\ln\delta + \delta^3\ln^2\delta + \delta^3\ln\delta + \mathrm{O}(\delta^4\ln^2\delta). \tag{19}$$

Using Eq. (6), one calculates the condensate depletion by means of the substitution (12) upon integrating

$$\frac{n-n_0}{n} = \int \frac{d^2q}{(2\pi)^2}n_q = u+\cdots. \tag{20}$$

Thus, with the help of Eqs. (15), (17), and (20), one can rewrite Eqs. (9) and (10) as

$$\mu = (4\pi\hbar^2n/m)u[1+u+\cdots], \tag{21}$$

$$g(r) = [\varphi^{(0)}(r)]^2 4u^2[1+2u+\cdots]. \tag{22}$$

Note that Eqs. (15) and (22) are the short-range approximation valid at $r\lesssim 1/\sqrt{n}$. For this reason, the boundary condition (2) is not fulfilled for $\varphi(r)$ in Eq. (15). In order to obtain the energy per particle ε, we represent it in the form $\varepsilon = (2\pi\hbar^2n/m)f(u)$ with an unknown function $f(u)$. From Eq. (18) it follows that $n\partial u/\partial n = u^2/(1-u)$, which, together with the thermodynamic relation $\mu = \partial(\varepsilon n)/\partial n$, yields the differential equation $u^2\mathrm{d}f/\mathrm{d}u + 2(1-u)f = 2u(1-u^2)$. Using the initial condition $f(u=0)=0$, we derive

$$\varepsilon = (2\pi\hbar^2n/m)[u+u^2/2+\mathrm{O}(u^3)]. \tag{23}$$

With Eq. (19), ε can be expanded in the parameter δ

$$\begin{aligned}\varepsilon = (2\pi\hbar^2n/m)[&\delta+\delta^2\ln\delta+\delta^2/2+\delta^3\ln^2\delta+2\delta^3\ln\delta\\ &+\mathrm{O}(\delta^3)].\end{aligned} \tag{24}$$

One can see that all the low-density expansions are series in the dimensionless in-medium scattering amplitude u, which depends ultimately on the density via Eq. (18) [and, hence, Eq. (19)]; therefore, u can be considered as *a parameter of low-density expansions in two dimensions*. The interaction energy per particle is exactly related to the pair distribution function (22)

$$\varepsilon_{\mathrm{int}} = \frac{n}{2}\int d^2rV(r)g(r) = \frac{2\pi\hbar^2n\alpha}{m}[u^2+2u^3+\cdots], \tag{25}$$

where we put by definition

$$\alpha = \frac{m}{\pi\hbar^2}\int d^2r[\varphi^{(0)}(r)]^2V(r) = \frac{2}{a}\frac{\partial a}{\partial\lambda}. \tag{26}$$

In the latter equation, we employ the theorem (A4) with the coupling constant λ [$V(r)\to\lambda V(r)$, and $\lambda=1$ in final formulas]. Since $\varepsilon_{\mathrm{int}}$ can be directly evaluated via the relation (25), our approach takes accurately into account the short-range particle correlations [13]. Note that Eq. (25) can also be obtained from the Hellmann-Feynman theorem $\varepsilon_{\mathrm{int}} = \lambda\partial\varepsilon/\partial\lambda$ with Eqs. (18), (23), and (26). Moreover, Eq. (22) can be derived in the same manner varying the energy: $g(r) = (2/n)\delta\varepsilon/\delta V(r)$. For the kinetic energy per particle $\varepsilon_{\mathrm{kin}} = \langle\Sigma p_i^2\rangle/(2mN)$ we have

$$\varepsilon_{\mathrm{kin}} = \varepsilon-\varepsilon_{\mathrm{int}} = \frac{2\pi\hbar^2n}{m}\left[u+\left(\frac{1}{2}-\alpha\right)u^2+\cdots\right]. \tag{27}$$

It is seen from Eqs. (23), (25), and (27) that in the leading order, proportional to nu, the total energy is purely kinetic. Thus, whatever a particular shape of the potential $V(r)$, at sufficiently small densities the energy becomes mostly kinetic. By contrast, in three dimensions $\varepsilon_{\mathrm{kin}}\simeq 2\pi\hbar^2bn/m$ and $\varepsilon_{\mathrm{int}}\simeq 2\pi\hbar^2(a-b)n/m$ are of the same order, where a is the 3D scattering length and $b = a-\lambda\partial a/\partial\lambda$ [13] (except for hard spheres when $a=b$, see Ref. [14]).

Let us discuss the nature of the Schick approximation $\varepsilon\simeq -2\pi\hbar^2n/(m\ln na^2)$, which is Eq. (24) in the lowest order in $n/\ln na^2$. As the energy in this order is purely kinetic, it cannot, in principle, be represented as a sum of the interaction energies of two particles over all pairs of bosons by analogy with the weak-coupling 3D Bose gas. However, as in three dimensions, we can start from Eq. (4) and put $\mu \simeq n\int \mathrm{d}^2rV(r)\varphi(r)$ with the *in-medium* pair wave function $\varphi(r)$. It is clear that $\varphi(r)$ for $r\lesssim r_0$ (here $r_0\sim 1/\sqrt{n}$ is of the order of the correlation length) should be proportional to the wave function $\varphi^{(0)}(r)$ of the two-body problem (A1): $\varphi(r) \simeq C\varphi^{(0)}(r)$. The boundary condition (2) can be fulfilled only *due to in-medium effects* for $r \gtrsim r_0$; therefore, we can approximately put $\varphi(r_0) \simeq 1$ and use for $r_0 \gg a$ the asymptotic (A2). This leads to $C \simeq -2/\ln na^2$ and, by Eq. (A3), yields $\mu \simeq -4\pi\hbar^2 n/(m \ln na^2)$ and, hence, the Schick result for ε. The crucial difference in the 3D case is the boundary condition for the two-body problem $\varphi^{(0)}(r) \to 1 - a/r$ instead of Eq. (A2). In this case, the condition $\varphi(r_0) \simeq 1$ leads to $\varphi(r) \simeq \varphi^{(0)}(r)$ in the leading order, which results in $\mu \simeq n\int \mathrm{d}^3 r V(r)\varphi^{(0)}(r) = 4\pi\hbar^2 na/m$, and, hence, $\varepsilon \simeq 2\pi\hbar^2 na/m$.

Now one easily writes 2D Gross-Pitaevskii functional for the energy using Eq. (23) in the leading order

$$E[\phi] = \int \mathrm{d}^2 r \left(\frac{\hbar^2 |\nabla \phi|^2}{2m} + V_{\mathrm{ext}}(\mathbf{r})|\phi|^2 + \frac{2\pi\hbar^2}{m} u |\phi|^4 \right), \tag{28}$$

and 2D Gross-Pitaevskii equation

$$\begin{aligned} i\hbar\partial\phi/\partial t &= \delta E/\delta\phi^* \\ &= [-(\hbar^2/2m)\nabla^2 + V_{\mathrm{ext}}(\mathbf{r})]\phi + (4\pi\hbar^2 u/m)|\phi|^2\phi. \end{aligned} \tag{29}$$

Here, $\phi = \phi(\mathbf{r},t) = \langle \hat{\psi}(\mathbf{r},t) \rangle$ is the order parameter with the normalization $N = \int \mathrm{d}^2 r |\phi|^2$, and u is given by Eq. (18) with $n = |\phi|^2$. Upon varying in Eq. (29), we neglect the variation of u, for $|\phi|^4 \delta u/\delta\phi^* \sim u^2 |\phi|^2 \phi$. Note that the sum of the first and third terms in Eq. (28) corresponds to the *kinetic energy* of bosons according to Eq. (27). Equations (28) and (29) are more exact than those of Ref. [15], based on the Schick formula.

In conclusion, the expansions have been derived for the condensate depletion (20), the chemical potential (21), the pair distribution function (22) for $r \lesssim 1/\sqrt{n}$, the total (23) [and (24)], interaction (25) and kinetic (27) energies. The energy expansion (23) leads to the Gross-Pitaevskii functional (28) and equation (29).

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APPENDIX

In this appendix, a useful variational theorem is proved for 2D scattering length. Here we deal only with the short-range potentials that go to zero for $r \to \infty$ as $V(r) \to 1/r^m$ $(m > 2)$, or even faster. The *s*-wave function corresponding to the relative motion of two particles with $p = 0$ obeys the two-body Schrödinger equation in the center-of-mass system

$$-(\hbar^2/m)\nabla^2 \varphi^{(0)}(r) + V(r)\varphi^{(0)}(r) = 0 \tag{A1}$$

with the following boundary conditions: first, $|\varphi^{(0)}(r)| < \infty$ at $r = 0$, and, second, for $r \to \infty$

$$\varphi^{(0)}(r) \to \ln(r/a). \tag{A2}$$

Since the asymptotic is chosen to be real, the solution of Eq. (A1) is also real. The introduced positive quantity a is called 2D scattering length. Integrating Eq. (A1) and keeping in mind Eq. (A2) yield

$$2\pi\hbar^2/m = \int d^2 r \, V(r)\varphi^{(0)}(r). \tag{A3}$$

Let us suppose that $V(r)$ is infinitesimally changed. Then, varying Eq. (A1), multiplying the obtained equation by $\varphi^{(0)}(r)$, and carrying out the integration, one arrives at the theorem

$$\frac{2\pi\hbar^2}{m}\frac{\delta a}{a} = \int d^2 r [\varphi^{(0)}(r)]^2 \delta V(r). \tag{A4}$$

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